Identification and rejection of waffle modes in layer-oriented adaptive optics

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ABSTRACT

Waffle modes are known in adaptive optics as wavefront perturbations giving a null or very small measurement on the wavefront sensor; when these modes appear on the deformable mirror, they are allowed to grow arbitrarly, generating loop instability and degrading the correction. In Multi-Conjugate Adaptive Optics, apart from the invisible modes related to the properties of the wavefront sensors, additional waffle modes may arise depending on the spatial configuration of the guide stars. The relevance of the problem to the layer-oriented approach is discussed, along with a practical method to identify them for a given guide stars asterism. Furthermore some strategies are described to face the problem at the real-time software level, with no additional complication on the hardware side.

Keywords: Waffle Modes, Multi-Conjugate Adaptive Optics, Layer-Oriented

1. INTRODUCTION

In an astronomical Adaptive Optics (AO) system there might some particular shapes generated by the Deformable Mirror (DM) that give null or very small signal onto the WaveFront Sensor (WFS). A Shack-Hartmann WFS with Fried geometry[1] for instance has waffle modes which might be eliminated adopting an altenative geometrical configuration[2]. Once they appear on the DM these modes may grow arbitrarily in amplitude, while the DM itself departs from the optimal shape, leading the system to instability and degrading the correction.

In a modal control framework[3] an effective method to cope with this problem relies on the inversion of the optical interaction matrix by the well-known Singular Value Decomposition (SVD) technique[4]. The interaction matrix establishes a link between the WFS and the DM; its generalized inverse is used as control matrix to convert the real-time slope measurements into DM commands. The null or very small singular values of the interaction matrix are associated to the low-sensitivity modes of the WFS[5]. When the matrix is inverted by the SVD technique the corresponding singular values in the generalized inverse are set to zero[6]. This has the effect of constraining the mirror shape in a way to cancel the low-sensitivity modes, hence preventing their arbitrary amplification.

The problem of waffle modes, presented so far for single-reference AO, has an analog in Multi-Conjugate Adaptive Optics[7] (MCAO). Here the wavefront measurements provided by several reference stars are combined to reconstruct the 3-dimensional structure of the turbulence and correct it by two or more DMs, usually conjugated to the stronger atmospheric layers, to achieve a more uniform correction across the Field of View (FoV). When the reference sources are arranged in a regular asterism there might be wavefront perturbations for which all the reference stars provide the same signal, so that it is not possible to determine the range at

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which the perturbation is occurring (Figure 1). These particular perturbations, treated in this paper, should be intended as invisible modes in a MCAO sense: although they do produce a signal on the WFSs, they do not allow a proper reconstruction of the 3-dimensional structure of the atmosphere. Again these modes may cause instability and correction degradation[8].

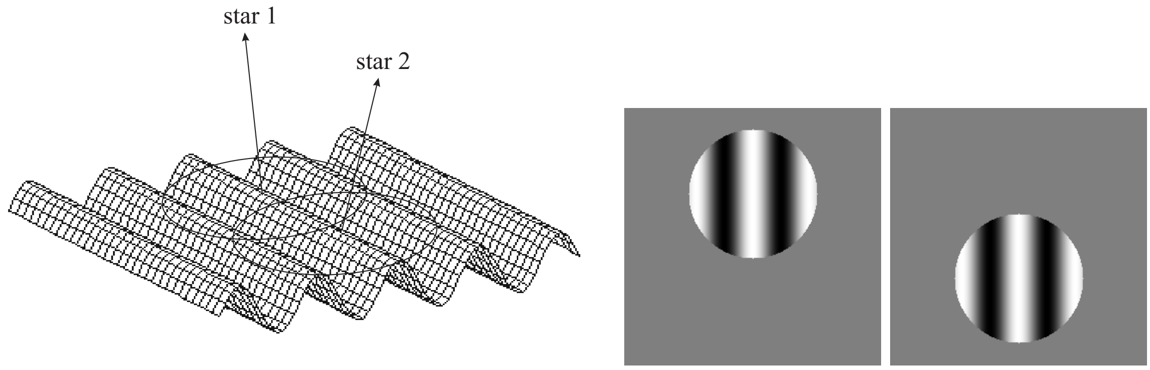

Figure 1. Left: a sinusoidal wavefront perturbation is sampled by two guide stars lying on the plane perpendicular to the perturbation. Right: the two stars experience the same wavefront aberrations and provide the same signal.

Different implementations of the MCAO concept have been described so far. In the global reconstructor approach[9] each WFS is coupled to a guide star and the signals are used to compute the commands for the DMs through a global control matrix. The global reconstructor may be optimized with a Maximum-A-Posteriori (MAP) technique[10], to include the statistical knowledge of the noise and of the atmospheric turbulence properties. A third implementation is the layer-oriented approach[11]; here each DM is driven by a separate detector focused on the same atmospheric slab and the wavefront reconstruction is accomplished by the optical co-addition of the reference signals. While a numerical implementation of layer-oriented is possible, with additional flexibility in the wavefront reconstruction, we limit here to the optical approach, in which the system is composed of independent AO loops, at least from the hardware point of view. In the following we derive the waffle modes for a two DMs layer-oriented system, we discuss the relevance of the problem under different conditions and we present some strategies for the real-time filtering of these modes.

2. WAFFLE MODES COMPUTATION

A detailed analysis of the layer-oriented approach has been presented elsewhere[8] and we just recall here the basic results. A layer-oriented system is composed of two or more independent AO loops, each one including a detector and a DM conjugated to the same atmospheric layer. The detector measures the wavefront error due to the whole atmosphere; however while the conjugated layer is seen in focus all the other layers are seen smoothed and the amount of blurring increases with the FoV and the distance of the considered layer from the plane where the detector is conjugated. In a 2-loops system, for instance, the detector no. 1, conjugated to the altitude h_1, sees the $l-th$ layer, located at the altitude h_l, smoothed by a convolution with the kernel

$$\delta_{1,l}\left(\mathbf{x}\right)=\sum_{s=1}^{S}\frac{I_s}{I}\delta\left[\mathbf{x}+\left(h_l-h_1\right)\theta_s\right] \tag{1}$$

where I_s is the intensity of the $s-th$ star ($s=1,\ldots,S$), I the total intensity of all the references, θ_s a 2-component vector identifying the position of the star, $\mathbf{x}$ a position vector on a plane orthogonal to the optical axis and δ the Dirac-delta. In practice the wavefront measurement for the considered layer is a sum of shifted copies of the layer itself, one for each star, and each copy is weighted by the relative intensity of the corresponding star. A relation similar to Equation 1 holds for the detector no. 2, conjugated to the altitude h_2. Incidentally we notice that this formulation does not account for the effect of the finite footprint size.

The analysis of the layer-oriented system may be conveniently carried out in the Fourier plane. It has been shown[8] that each spatial frequency evolves independently in time. There are frequencies, depending on the spatial distribution and brightness of the reference stars, for which the layer-oriented system is not stable, hence subject to noise amplification and lack of convergence. Formally it may be shown that for these frequencies the response to an impulse input does not tend to zero as it should and the response to a step input increases arbitrarly with time. From the wavefront sensing point of view these frequencies do not allow a proper 3-dimensional reconstruction of the atmospheric turbulence. It is therefore natural to associate them to the waffle modes of the layer-oriented system. In the simple though significative case of two DMs it has been shown that the waffle frequencies are identified by the condition

$$|\tilde{\delta}_{1,2}| = 1 \tag{2}$$

where $\tilde{\delta}_{1,2}$ is the Fourier transform of the functional in Equation 1 when the two conjugated planes are considered, namely

$$\tilde{\delta}_{1,2}(\mathbf{u}) = \sum_{s=1}^{S} \frac{I_s}{I} e^{i2\pi(h_2-h_1)\theta_s \cdot \mathbf{u}}. \tag{3}$$

The simplest invisible frequency is $\mathbf{u} = 0$, corresponding to the piston mode. Any other waffle mode is identified by a couple of frequencies symmetric with respect to the origin, namely the points (u_i, v_i) and $(-u_i, -v_i)$. The shape of the waffle mode in the direct space results to be a sinusoid with spatial frequency $f_i = \sqrt{u_i^2 + v_i^2}$. When also the frequencies in the neighborhood of the points (u_i, v_i) are taken into account, the corresponding shape is an amplitude-modulated sinusoid.

Not all the waffle frequencies are of practical relevance: the finite number of degrees of freedom of the DMs, in fact, imposes a kind of spatial cutoff which prevents the amplification of the high-frequency modes, simply because they cannot be reproduced by the DM. Assuming an inter-actuator distance d, the shortest wavelength that can be generated is of the order of $2d$, which corresponds to a cutoff frequency

$$f_c \approx \frac{1}{2d}. \tag{4}$$

The functional in Equation 3 has been computed here for different guide star asterisms, characterized by equally bright sources for simplicity (Figure 2). A threshold smaller than unity has been defined for the identification of the peaks of the functional, representing the waffle (or quasi-waffle) frequencies in the Fourier plane. For a trivial asterism with only two reference stars the waffle frequencies are represented by parallel lines. In a Natural Guide Stars (NGS) asterism with randomly spread sources the number of waffle frequencies is always lower than for a regular asterism with the same number of sources. Furthermore the number of waffle modes tends to decrease as the number of stars increases. For an asterism formed by several reference stars, for instance $S = 8$, the number of waffle frequencies is usually negligible and in most cases only the piston remains. It should also be stressed that with an irregular configuration most of the peaks of the functional are not equal to unity, i.e. they do not fulfill exactly the condition expressed by Equation 2. The modes associated to these frequencies are not invisible in a strict sense, however it is interesting to investigate their effect on the system behaviour. The conclusion of this discussion is that the problem of waffle modes in a practical implementation of the layer-oriented approach[12] is likely to have some relevance only for asterisms composed of few reference stars, as those that might be found at the Galactic Pole for instance.

3. WAFFLE MODES REJECTION

Since the system is not able to properly reconstruct the waffle modes, the most obvious thing to do (and perhaps the only one) is to remove the waffle component from the applied correction. It should be stressed that the waffle frequencies present in the atmospheric turbulence are not corrected at all in this way: this is the price to be paid to ensure the system stability and convergence. In the global reconstructor method the control matrix may be modified to project the computed command vector onto a subspace orthogonal to that generated by the low-sensitivity modes[9]. In the MAP-based approach[10] the regularization inherent to this technique should

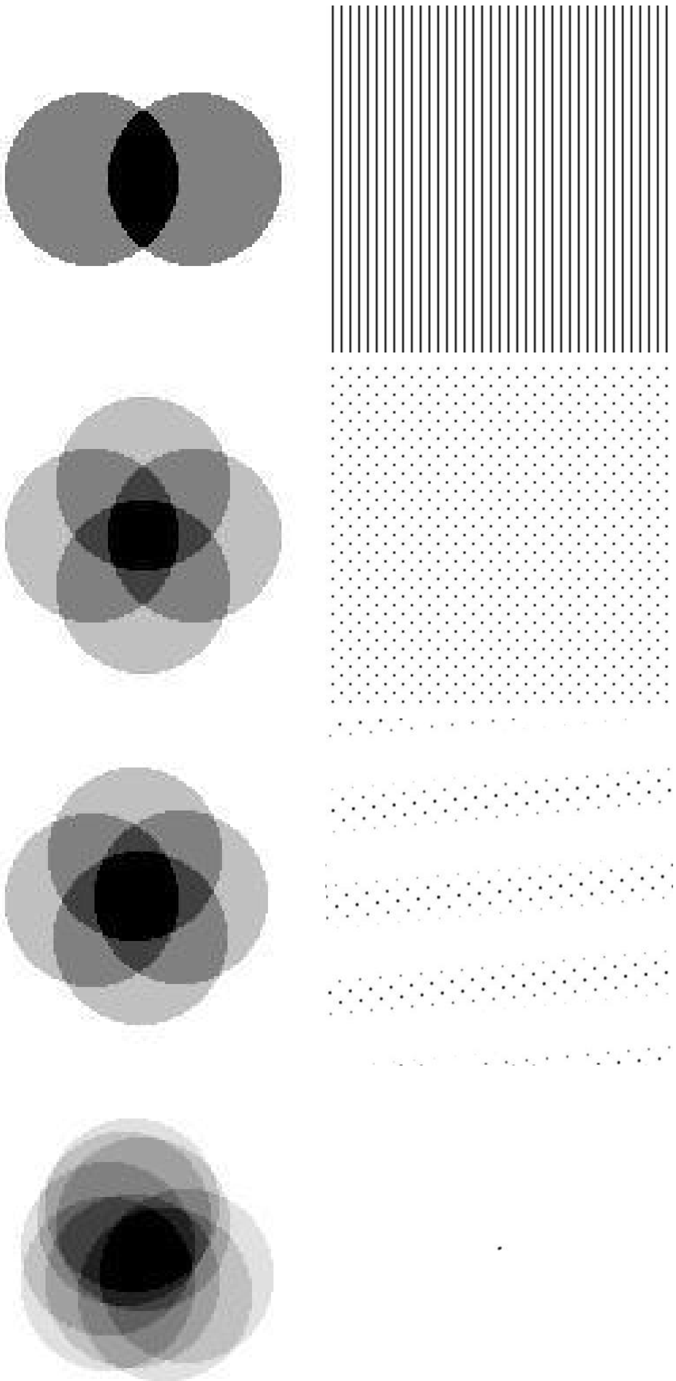

Figure 2. Left: footprints for different guide stars asterisms, corresponding to a conjugation altitude of 8.5km for a 8m telescope on a 2′ FoV. Right: waffle frequencies in the Fourier plane; the center of the arrays corresponds to the zero frequency. The cases shown here correspond, from top to bottom, to 2 stars, 4 stars arranged in a regular square, 4 stars in a slightly irregular configuration and 8 stars in a random asterism.

prevent the arbitrary amplification of the waffle modes. In the layer-oriented approach the filtering may be similarly accomplished by modifying the control matrix of each independent AO loop forming the system.

In the following we describe two different methods to suppress the waffle modes from the applied correction in a layer-oriented system: one is a kind of regularization of the control matrix, the other is based on a projection approach. Both these method are formulated in a modal control framework, assuming as basis modes for each DM a set of orthogonal (e.g. Zernike) polynomials $Z_n(x,y)$, where the maximum number N may be different from DM to DM. In general none of the mirror modes, considered separately, is a waffle. However certain combinations of polynomials might be very close to some waffle mode. In each independent loop forming the layer-oriented system, the link between the WFS and the DM is established by an interaction matrix $\mathbf{M}$ whose columns are the slopes measurements corresponding to each DM mode, applied with unit amplitude. The relation between a given command vector $\mathbf{c}$ and the corresponding slope vector $\mathbf{s}$ is given by

$$\mathbf{Mc} = \mathbf{s}. \tag{5}$$

The interaction matrix is assumed to be inverted by the SVD technique to compute the control matrix $\mathbf{M}^+$ used in closed loop to convert the slope measurements into DM commands. An interesting property of the SVD method, already mentioned in Sec. 1, is the possibility to set a cutting threshold for the small singular values of the matrix to be inverted; this singular values are set to zero in the generalized inverse. A convenient way to define this threshold is in terms of the condition number Q, defined as the ratio of the maximum to the minimum (in absolute terms) non-zero singular value. Adopting a condition number $Q = 10$, for instance, it is possible to decide which singular values should be discarded and set to zero in the SVD inversion.

3.1. Constrained inversion of the interaction matrix

The first method to prevent waffle modes amplification is based on a modification of the control matrix $\mathbf{M}^+$ of each independent loop forming the layer-oriented system. From the modal expansion of the correction

$$c(x,y) = \sum_{n=1}^{N} c_n Z_n(x,y) \tag{6}$$

it is clear that the $i-th$ waffle mode (u_i, v_i) is removed if the Fourier transform of the correction fulfills the conditions

$$\sum_{n=1}^{N} c_n \tilde{Z}_n(u_i, v_i) = 0 \tag{7}$$

where $i = 1, \ldots, I$ is an index for the waffle frequencies of interest and $\tilde{Z}_n$ is the Fourier transform of the *n-th* mirror mode. These conditions represent linear constraints on the coefficients of the modal expansion of the correction. They can be expressed as

$$\mathbf{z}_i^T \mathbf{c} = 0. \tag{8}$$

Including these constraints, the inverse problem in Equation 5 becomes

$$\begin{cases} \mathbf{Mc} &= \mathbf{s} \\ \mathbf{Zc} &= \mathbf{0} \end{cases} \tag{9}$$

where the matrix $\mathbf{Z}$ is formed by the row vectors $\mathbf{z}_i^T$. The previous system of equations is an overdetermined one and can be solved only in a least-squares sense. A possible solution is the regularized one

$$\mathbf{c} = \left(\mathbf{M}^T\mathbf{M} + \alpha\mathbf{Z}^T\mathbf{Z}\right)^{-1}\mathbf{M}^T\mathbf{s} \tag{10}$$

where α is a free parameter which establishes a trade off between goodness of fit and the constraint represented by the waffle modes suppression. The matrix multiplying the slope vector $\mathbf{s}$ on the right-hand side of Equation 10 is the control matrix for the closed loop operation. Since the size of the control matrix does not increase, the computation time in closed loop is not affected. The main drawbacks of this technique are represented by the additional free parameter α and by the fact that the waffle modes are not suppressed perfectly.

3.2. Subtraction of the waffle mode component

A second possibility is to remove from the computed command vector the component parallel to the subspace generated by the waffle modes; this is equivalent to projecting the command vector onto the subspace orthogonal to that generated by the waffle modes[9] (Figure 3). In the simplest case of just one waffle mode $w_1(x, y)$, the orthogonal component is

$$\mathbf{c}_\perp = \mathbf{c} - \mathbf{w}_1 \left(\mathbf{w}_1^T \mathbf{c}\right) = \left(\mathbf{I} - \mathbf{w}_1 \mathbf{w}_1^T\right) \mathbf{c}, \tag{11}$$

where $\mathbf{w}_1$ is a normalized vector formed by the components of the waffle mode onto the mirror basis. When the number of relevant waffle modes is $I > 1$, a straightforward generalization of the method consists of decomposing each waffle mode onto the mirror basis, thus obtaining a set of component vectors. These are then ortho-normalized, in order to find an orthonormal basis $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_I$ for the linear subspace of the waffle modes. Then the DM command vector orthogonal to this subspace may be computed as

$$\mathbf{c}_\perp = (\mathbf{I} - \mathbf{W})\,\mathbf{c} \tag{12}$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{W}$ is defined as

$$\mathbf{W} = \sum_i \mathbf{w}_i \mathbf{w}_i^T. \tag{13}$$

The projection operator onto the subspace orthogonal to that generated by the waffle modes may also be multiplied by the control matrix $\mathbf{M}^+$, obtaining the modified reconstructor

$$\mathbf{M}_\perp^+ = (\mathbf{I} - \mathbf{W})\,\mathbf{M}^+ \tag{14}$$

which gives only command vectors orthogonal to the waffle subspace. The modification of the control matrix has to be done for each independent loop separately, as for the technique presented before. Even the present method does not increase the real-time computional demands, because the size of the control matrix is left unchanged.

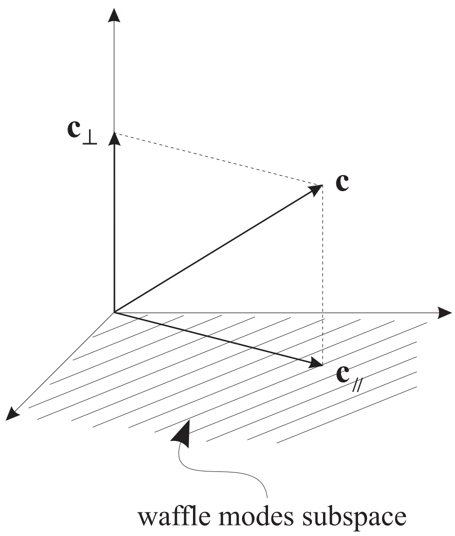

Figure 3. Waffle modes filtering is accomplished by projecting the computed command vector onto the subspace of admissible mirror modes orthogonal to the waffle modes subspace. Here the space of all command vectors is pictorially represented by a 3D vector space, where the XY plane coincides with the waffle modes subspace.

4. NUMERICAL SIMULATIONS

According to the discussion presented in Sec. 2, the problem of waffle modes amplification might be relevant with NGS asterisms composed of few reference sources. For this reason we have performed a set of simulations with asterisms of $S = 4$ guide stars. We have investigated for completeness both the case of a regular asterism (four stars at the vertex of a square), although this is a rather unrealistic condition, and five irregular asterisms, in which the stars positions have been randomly perturbed by at most $10 - 15''$. In all the cases the reference sources are spread over a FoV of approximately $2'$ in diameter. For simplicity we have assumed equally bright stars, with brightness $mag = 12$ or $mag = 15$. Both photon noise and instrumental noise, due to read-out and dark current, have been included. The amount of noise is representative of MAD, a MCAO prototype currently under development[12]. In order to avoid any anisoplanatic effect, we have considered a simplified atmospheric model with only two layers placed at the same ranges where the two DMs are conjugated. The total power of the turbulence, however, corresponds to a realistic situation of a seeing with FWHM $\approx 0.9''$ in the visible. The two detectors have been sampled with 10×10 sub-apertures each, so that a sub-aperture is slightly smaller than the r_0 of each layer. No temporal evolution of the layers has been introduced: the main goal of the tests was to verify the capability of the system to converge to a static input turbulence, evaluating the impact of the waffle frequencies. Finally we have implemented two wavefront reconstruction approaches: zonal and modal. In the former one, the re-binned phase maps provided by the WFSs have been interpolated onto a finer grid and applied to the corresponding DMs, without any interaction matrix. In the modal case the number of Zernike polynomials has been set to 72 and 68 modes for the two DMs respectively, i.e. approximately 10% less than the maximum number of modes compatible with the number of available sub-apertures; the control matrix has been computed by the SVD inversion of the interaction matrix formed by the Zernike modes, adopting a reasonable figure of $Q = 10$ on the maximum allowed ratio between the maximum and minimum singular value. It should be stressed that this figure has been adopted as a reasonable one, with no optimization. With this threshold, the number of selected singular values is 57 for the ground layer and slightly smaller for the high altitude layer, the exact value depending on the guide stars asterism.

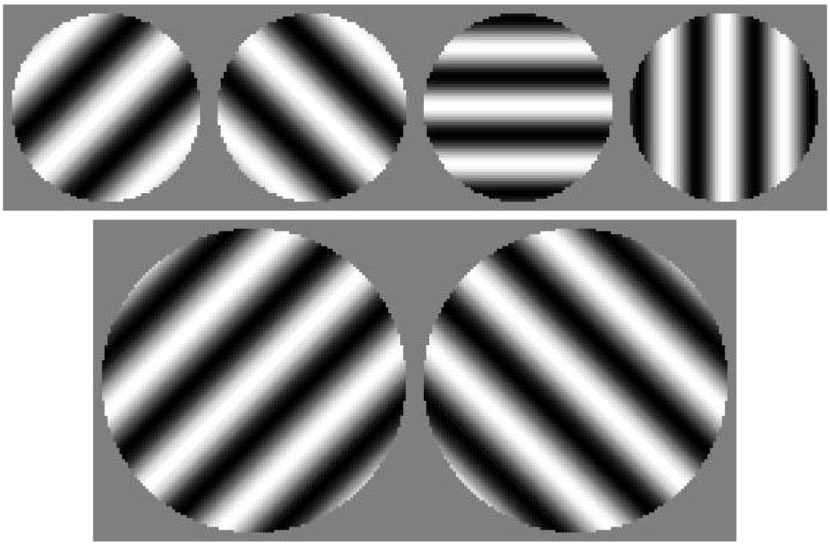

Figure 4. The relevant waffle modes for the ground and the high-altitude conjugated DMs for the regular asterism of four stars considered in the simulations.

The relevant waffle modes for the adopted spatial sampling in the case of the four stars regular asterism are shown in Figure 4. The number of waffles modes is lower for the high-altitude DM because of the coarser spatial sampling (same number of sub-apertures on a larger region). In the irregular asterisms the number of modes below the cutoff frequency ranges from 1 to 4; it should also be noticed that in these cases, apart from the zero frequency associated to the piston, the other waffle frequencies below the cutoff are not invisible in a strict sense, according to the discussion of Sec. 2.

4.1. Zonal case

In the first tests we have adopted the zonal wavefront reconstruction approach. The results (Figure 5) indicate that both with regular and irregular guide stars asterisms the correction at the center of the FoV, where there is no reference source, is worse than on the four guide stars. Since the DMs are conjugated to the layers, the Strehl Ratio (SR) should be rather uniform across the FoV and the SR homogeneity should be limited only by the uniformity of the photon density in the high altitude layer. We interpret this SR dishomogeneity as a signature of the waffle frequencies, that prevent the system to converge and provide a useful correction. To confirm further this hypothesis, we have repeated the same test with an asterism formed by 8 randomly spread stars, scaling their brightness in order to have the same integrated magnitude, and we have experienced no SR degradation across the field as that illustrated in Figure 5. The conclusion of this test is that the problem of waffle modes is relevant in a zonal approach when the asterism is formed by few reference stars, even in an irregular configuration, although we have observed cases with irregular asterisms in which the correction is not so poor as in Figure 5. We have not tried to implement any waffle control technique in the zonal case, switching to the modal approach, which seems to be more representative of real systems.

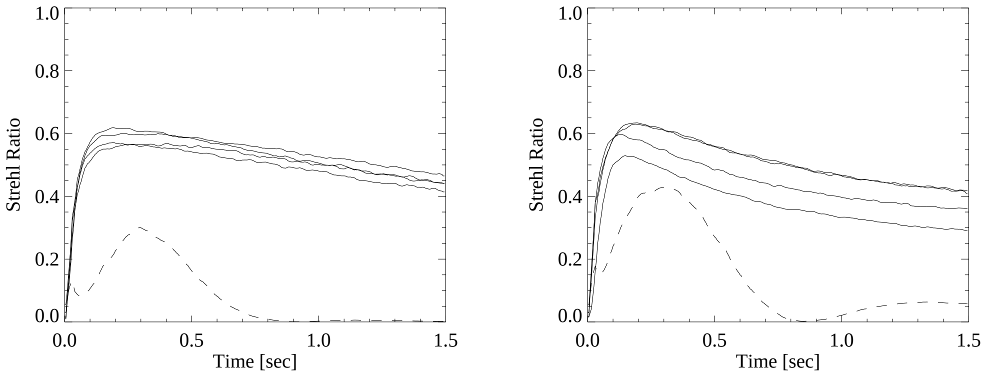

Figure 5. Zonal case: SR temporal evolution. The dashed lines represent the SR at the center of the FoV, where there is no guide star; the continuous lines refer to the reference sources. Two asterisms with 4 guide stars have been considered: a regular (left) and an irregular one (right).

4.2. Modal case

If the four stars asterism is regular, even with the modal approach the waffle modes are relevant (Figure 6, left). Although these tests are to be taken as an exercise, because of the regular stars configuration, we have tried to apply a preliminary implementation of the waffle control technique described in Sec. 3.2. In the first case presented in Figure 6 (right) the technique makes the system converge, achieving a good correction even at the center of the field. It is not so good as on the reference stars, but it should be stressed that the aim and capability of this method is just to remove the cause of instability, without assuring the optimality of the correction. In the second case presented in the Figure the effect of the waffle control technique is to improve the correction at the center of the FoV, which is not so bad even without waffle control.

The situation is completely different with the five irregular asterisms described before. An example of the achieved performance is shown in Figure 7: no waffle mode amplification problem has been experienced here and the correction is reasonably uniform across the FoV without any waffle control technique, both with bright and faint reference stars. The control matrix computed by the SVD with a reasonable condition number $Q = 10$ is sufficient to ensure the system convergence and stability.

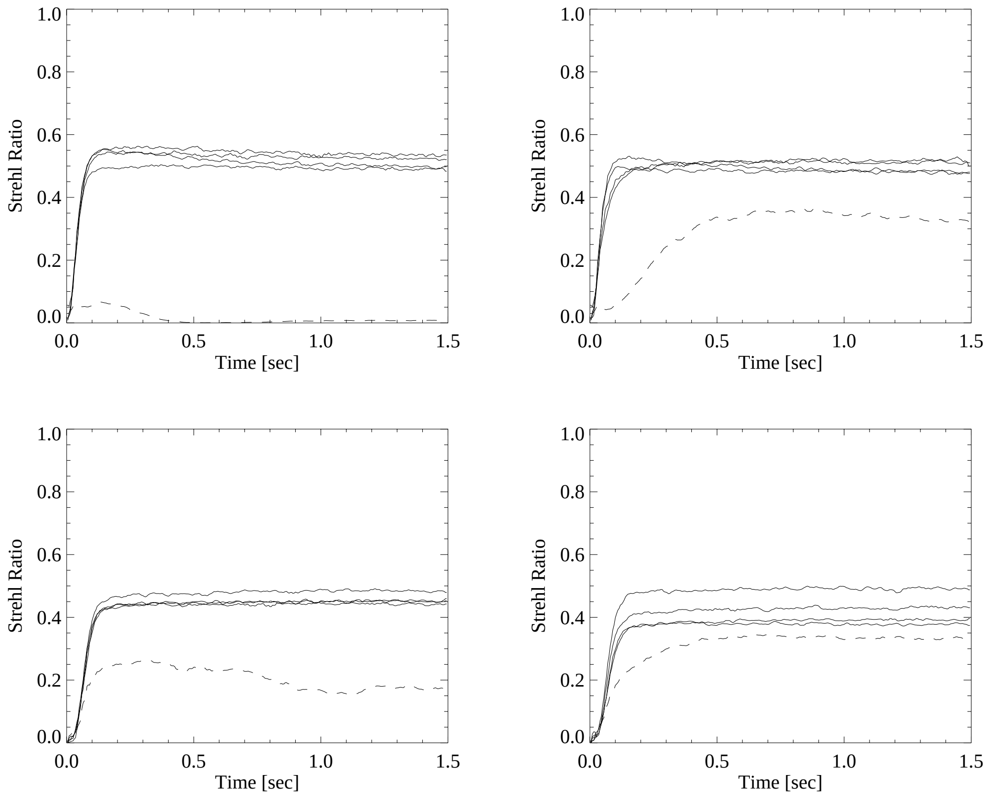

Figure 6. Modal case: SR temporal evolution for the regular asterism. The meaning of the dashed and continuous lines is the same as in the previous Figure. Left: no waffle control. Right: with waffle control.

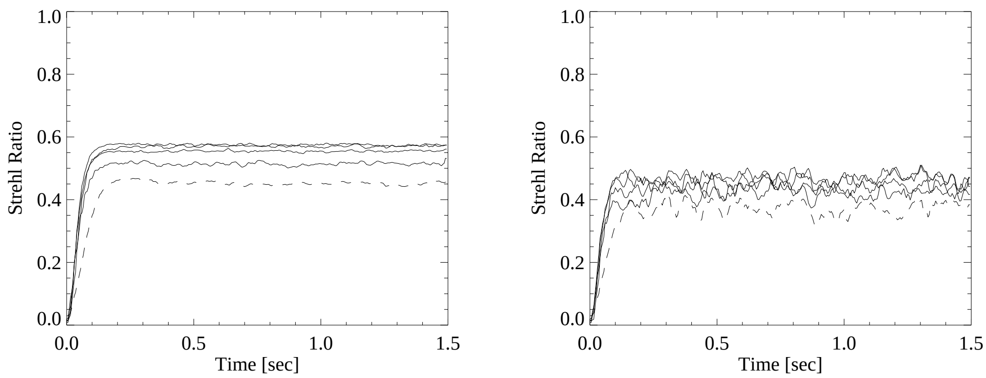

Figure 7. SR temporal evolution for an irregular asterism with modal wavefront reconstruction. The brightness of each guide star is $mag = 12$ (left) and $mag = 15$ (right).

5. CONCLUSIONS

We have investigated the effect of the waffle modes related to the spatial distribution of the reference stars in a layer-oriented MCAO system. We have shown that the problem might have a practical relevance only for asterisms composed of few reference sources and the case of four guide stars has been assumed as a representative one. In the zonal approach the problem of waffle modes with such kind of asterisms might be relevant. However in the most representative case of modal control we have experienced problems of instability and lack of convergence only with a regular asterism, a quite unrealistic condition in practice. For this particular case we have tested a filtering method to remove the waffle mode component from the computed DM commands, acting on each loop independently. This technique retains one of the fundamental properties of the layer-oriented approach, namely the fact to have independent AO loops with no global reconstructor. The method we have implemented has proven to be quite effective, although further investigation is required concerning the spatial cutoff frequency imposed by the DMs, the effect of the high-order waffle modes beyond the cutoff frequency and the behaviour of systems with finer spatial sampling than the one considered here.

The most relevant result of this work, however, is that with irregular guide stars asterisms, even with few reference stars, no problem of instability or correction degradation has been experienced with modal control. The application of the SVD technique with a suitable condition number yields a control matrix which is sufficient to prevent waffle modes amplification. This has been shown under different signal-to-noise ratio conditions, hence it seems to be a general result. This means that the problem of waffle modes in the forthcoming layer-oriented WFS for the ESO MCAO demonstrator[12] might be completely unrelevant or easily solvable with a judicious definition of the control matrix of each loop. In any case the investigation of the waffle control method will continue, in order to improve the effectiveness of the technique, also in view of possible future implementations.

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